AMENABLE ORBIT RELATIONS AND POINTWISE EQUIPARTITION BEYOND AMENABLE GROUPS

j/w in progress
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$(\Omega, \mathcal{F}, P)$ P-space, S finite set

$\Omega = S^{\mathbb{Z}}$, $\sigma : \Omega \to S$

$I(\sigma) : \Omega \to [0, \infty)$

$\omega \mapsto -\ln P(\{\tilde{\omega} \in \Omega : \sigma(\tilde{\omega}) = \sigma(\omega)\})$

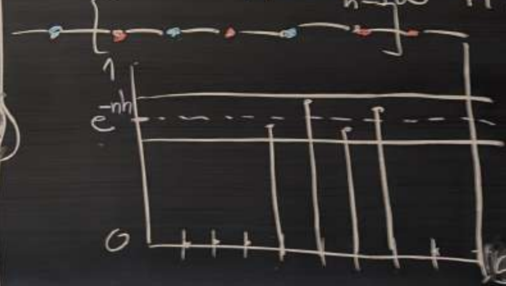
$(\sigma_n)_{n \in \mathbb{Z}}$, $\sigma_n : \Omega \to S$ stationary ergodic

Theorem (Shannon-McMillan-Breiman) SMB

There exists $h \in [0, \ln |S|]$ s.t.

$$\frac{1}{n} I(\sigma_1 \dots \sigma_n) \xrightarrow{n \to \infty} h \quad P\text{-a.s.}$$

$$h = \lim_{n \to \infty} -\frac{1}{n} \sum_{w \in S^n} P(w) \ln P(w)$$



asymptotic equipartition property

$\mathbb{Z} \rightsquigarrow$ ctb. group Γ

$(\sigma_\gamma)_{\gamma \in \Gamma}$ Stat. and ergodic

<u>Theorem</u> (Lindenstrauss '01, Weiss '03)

Γ amenable, $(F_n)_{n=1}^{\infty}$ tempered Følner sequence

Then there exists $h \in [0, \ln|S|]$ s.t.

$$\frac{1}{|F_n|} I(\sigma_{F_n}) \xrightarrow{n \to \infty} h \quad \mathbb{P}\text{-a.s.}$$

<u>Ex.</u>: ▷ $\Gamma = \mathbb{Z}^d$, $F_n :=$ cube of length n

▷ $\Gamma = \langle G \rangle$ f.g. polynomial volume growth

$\rightsquigarrow F_n :=$ balls of radius n

$\Gamma = \mathbb{F}$ free group over r generators S

$\Lambda, \Delta \subseteq \Gamma$ finite

$$\psi(\Lambda, \Delta) := \sup \left\{ \left| \frac{P(A \cap B)}{P(A)P(B)} - 1 \right| : A \in \mathcal{F}_\Lambda, B \in \mathcal{F}_\Delta, P(A)P(B) > 0 \right\}$$

<u>Def.</u>: $\lambda, C > 0$. $(\sigma_\gamma)_{\gamma \in \Gamma}$ is λ-exponentially ψ-mixing if for all Λ, Δ sufficiently large apart we have

Theorem (Shannon-McMillan-Breiman)

$$\psi(\Delta, \Lambda) \leq C \cdot |\Delta| |\Lambda| e^{-\lambda d(\Delta, \Lambda)}$$

<u>Ex.</u>: ▷ iid

▷ Gibbs measures (Ising models)

<u>Theorem</u>: $\lambda > \frac{3}{2} \ln 2r$; $(\sigma_\gamma)_{\gamma \in \mathbb{F}}$ stat. and λ-exp. ψ-mixing. Then there exists $h \in [0, \ln |S|]$

$$\frac{1}{|S_{2n}|} I(\sigma_{\underbrace{S_{2n}}_{= \{\gamma : |\gamma| = 2n\}}}) \longrightarrow h \quad P\text{-a.s.}$$

(X, E, μ) pmp CBER, ergodic

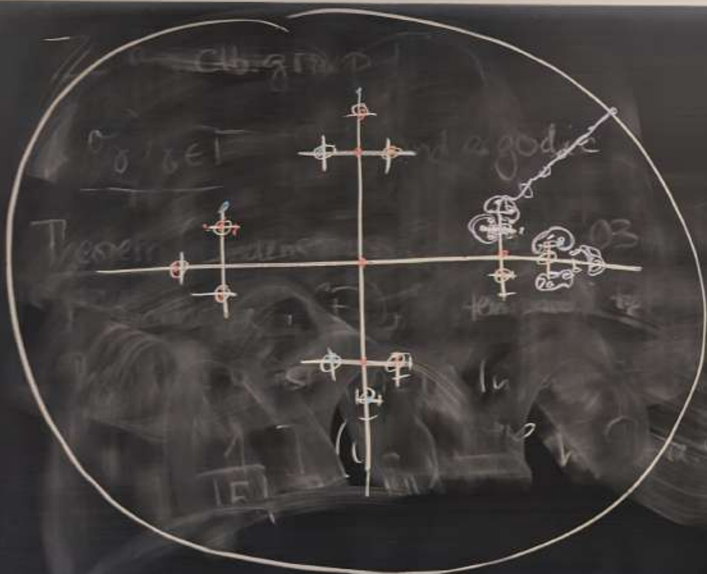
$\alpha : E \to \Gamma$ msb. cocycle

$F \subseteq E \rightsquigarrow F^\alpha : X \to \{\text{subsets of } \Gamma\}$

$x \mapsto \{\alpha(x,y) : (x,y) \in F\}$

<u>Theorem</u>: $(\sigma_\gamma)_{\gamma\in\Gamma}$ stat. and ergodic, Borel $\alpha : E \to \Gamma$ weakly mixing, $(F_n)_{n=1}^\infty$, $\underline{F_n \subseteq E}$, regular Følner sequence. Then $\exists\, h \in [0, \ln|S|]$

$$\frac{1}{|F_n^\alpha(x)|} I\left(\sigma_{F_n^\alpha(x)}\right) \longrightarrow h \quad \mu\otimes\mathbb{P}\text{-a.s.}$$



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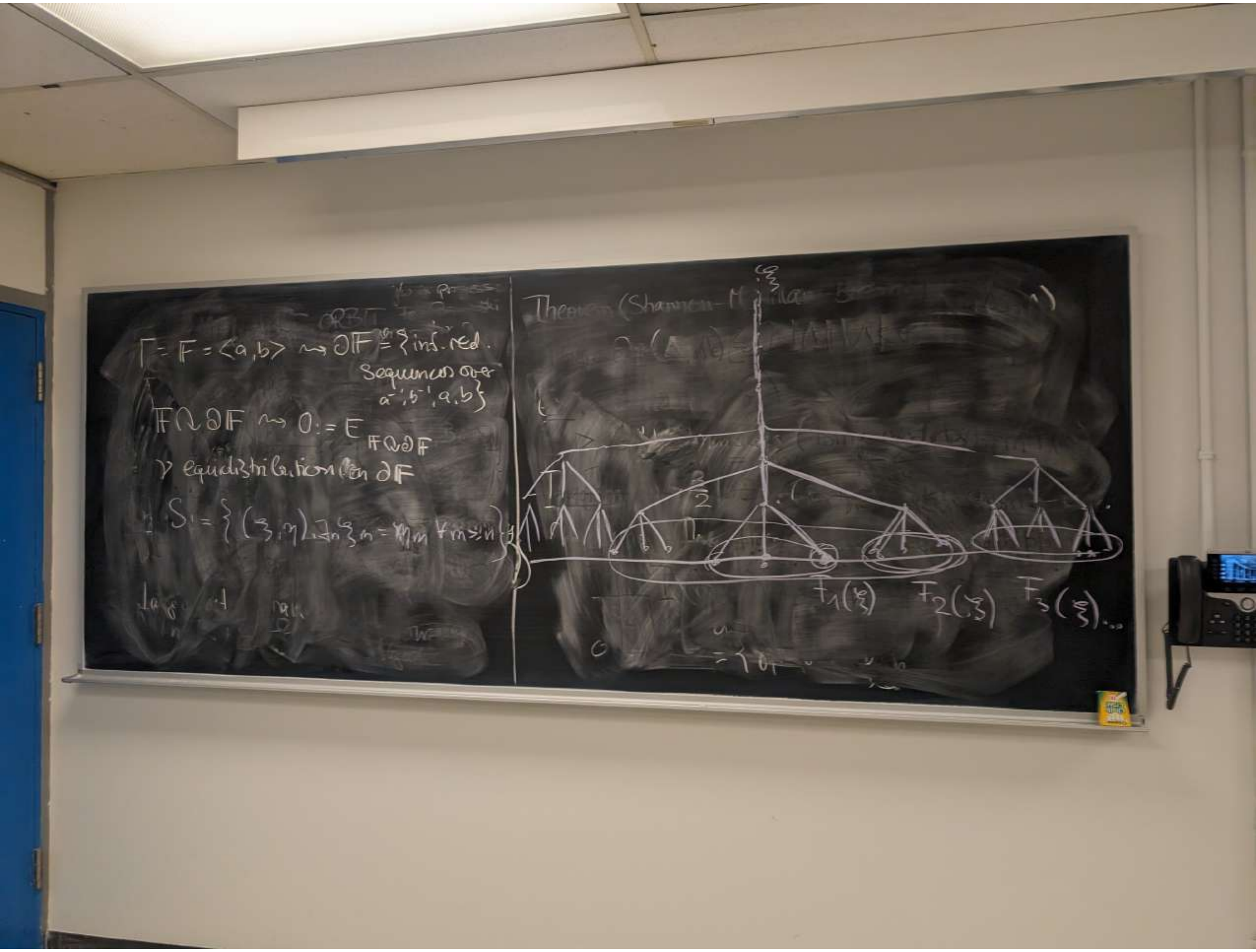

ORBIT
$\Gamma = \mathbb{F} = \langle a,b \rangle \rightsquigarrow \partial\mathbb{F} = \{$inf. red. sequences over $a^{-1}, b^{-1}, a, b\}$
$\mathbb{F} \curvearrowright \partial\mathbb{F} \rightsquigarrow O := E_{\mathbb{F} \curvearrowright \partial\mathbb{F}}$
ν equidistribution on $\partial\mathbb{F}$
Theorem (Shannon-
$F_1(\xi)$
$F_2(\xi)$
$F_3(\xi)\dots$